Pergamon

J. Appl. Maths Mechs, Vol. 58, No. 5, pp. 759–776, 1994

0021–8928/94 $24.00+0.00

0021–8928(94)00101-4

COMOVING AND INERTIAL FRAMES OF REFERENCE†

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(*Received* 29 *September* 1993)

The possibility of constructing the non-linear equations of the general theory of relativity (GTR) incorporating a potential energy concept is demonstrated. Solutions of the generalized equations are considered for the motions of individual particles with non-vanishing four-dimensional absolute acceleration relative to a family of inertial geodesics, which satisfy the field equations in their orbits. Unless the law of universal gravitation is explicitly taken into account, the equations of the GTR and the field equations do not constitute closed systems.

The law of universal gravitation imposes additional restrictions on the gravitational field and the trajectories of free particles. The description of the relativistic gravitational fields and the free motion of mass particles is based on using both the thermodynamic energy scalar mc^2 and the potential energy scalar mU of the particles, just as in Newtonian mechanics or in Minkowski space in the special theory of relativity (STR). In a comoving frame of reference the scalar U satisfies a three-dimensional Poisson equation.

In the light of the theory proposed here, many well-known solutions of the GTR have to be reinterpreted.

The construction of mathematical models using differential equations generally involves the adoption of various simplified "basic" axioms and laws that have been tested in various ways for infinitely small volumes in space and time. The latter are defined using suitable canonical systems of coordinates in special, ideal, geometrical kinematic objects that enable one to characterize the phenomena under study and to introduce various specific motions of a mathematical nature. Thus, in order to describe and predict the required answers one has only to formulate and solve mathematical problems, as a result of which the essence of the objects under consideration and their interactions can be understood, subject to the necessary approximations, and the nature of various events described.

On a theoretical level, the mathematical methods used to investigate the explanations and conclusions thus obtained by computational means frequently involve suitable systems of coordinates or frames of reference as investigative tools. These frames of reference may vary, but it is understood that the discovery of the basic reasons for their implications and the specific properties of the phenomena being studied must be invariant in some suitable sense dictated by the formulations of the problems and the means used to solve them.

The use of tensor methods enables one to find results that do not depend on the particular forms of the systems of coordinates adopted.

In definitions and when operating with space–time, material bodies and fields, one must work with continuous manifolds of "points" which are individualized by their properties, masses, charges, spins and in general many other specific features which come to light when they interact.

†*Prikl. Mat. Mekh.* Vol. 58, No. 5, pp. 3–21, 1994.

The individuality of the objects is postulated by stipulating notions about their properties and interactions. In particular, this applies to representations of space–time as manifolds of individual carriers with geometrically defined continua of points, embedded in which are various objects with their own individual points. It is taken for granted that such individuals—which may or may not be the same as the individuals of the spaces and time being used—interact.

In traditional physical applications, however, one uses four-dimensional physically defined spaces. In such spaces one attaches prime importance to space–time frames of reference, in which the arithmetization coordinates of individual points are produced by naming three constant values of the numbers ξ^1, ξ^2, ξ^3, and proper time τ. In a fixed space one can consider various objects, and along with the coordinates (ξ^1, ξ^2, ξ^3, τ) one can similarly introduce further objects with coordinates $\eta^i(\eta^\alpha, \tau')$, $(\alpha = 1, 2, 3)$. Thus, one can refer to comoving systems of coordinates for an observer $\eta^i(\eta^1, \eta^2, \eta^3, \tau')$ and for the substances being studied, which are defined by the coordinates ξ^α and τ, and speak of motion in one and the same space relative to an observer defined by the function $\eta^i(\xi^k)(i, k = 1, 2, 3, 4)$.

In various physical theories, numerical characteristics of "point-objects," designated by constant values $\xi^\alpha = \text{const}(\alpha)$, may be replaced by suitable objects using matrices of numbers and other generalized concepts. In particular, ξ^α may represent such large-scale finite bodies as asteroids, planets, stars and even whole galaxies. Not infrequently, such objects are modelled by material points with designated values of ξ^1, ξ^2, ξ^3 and τ.

There are different ways of defining systems of coordinates in a point continuum. Bearing in mind four-dimensional pseudo-Riemannian spaces with metrics having signature $---+$, families L of global time coordinates τ, corresponding to + values in the metric, may also be taken to correspond to coordinates lines L, where the latter were chosen fairly arbitrarily for the variable τ with $\xi^\alpha = \text{const}$ at the initial stage, in the construction of the theory—subject nevertheless to the restriction that they are not closed and, because of the uniqueness condition, do not intersect. In the final phase of investigation, one has to set conditions to define the possible families of such lines for Riemannian spaces. Clearly, when working in specific relativistic theories, one must put forward additional conditions, which should define systems of families of coordinate lines L representing the influence of certain characteristic quantities. The latter may arise and will generally figure in the axiomatic model relationships prescribed by the specific features of the local inertial tetrad reference systems being used.

It should be noted that in formulations of gravitational theory, which is devoted to the free, collisionless motion of material particles, one considers the mutual interaction of particles via mass forces, without any surface forces. The latter are characteristic, for example, in the theory of elasticity and plasticity, the theory of viscous fluids and many other (model) material and field continuous media.

Without loss of generality, the metric for individual particles, which are defined by constant values of ξ^1, ξ^2, ξ^3 and the variable τ, may be transformed globally [1] to canonical form

$$\begin{aligned} ds^2 &= \overset{0}{g}_{ij}(z^1, z^2, z^3, z^4)dz^i dz^j, \qquad i, j = 1, 2, 3, 4 \\ ds^2 &= c^2 d\tau^2 + 2g_{\alpha 4}(\xi^\gamma, \tau)d\xi^\alpha d\tau + g_{\alpha\beta}(\xi^\gamma, \tau)d\xi^\alpha d\xi^\beta \end{aligned} \tag{1}$$

where $\alpha, \beta = 1, 2, 3$ and the summation convention for equal indices is assumed. Here ds is the invariant distance between any suitably chosen infinitesimally close points in the space.

After any particular transformation we have $z^k = z^k(\xi^1, \xi^2, \xi^3, \tau)$, and c is a constant scalar characteristic of the pseudo-Riemannian space.

In this form of the metric, the family of L consists of coordinate lines of the variable τ on which $ds^2 = c^2 d\tau^2$ for individual points, the latter being designated by quantities $\xi^\alpha = \text{const}(\alpha)$.

In comoving Lagrange coordinates for a fixed Riemannian space with metric (1), all the mechanical characteristics of the state of individuals may be treated as particles embedded into the space. The state of a continuous medium is defined by functions of the components of the metric tensor $g_{44} = c^2 = \text{const}$, $g_{\alpha 4}(\xi^\alpha, \tau)$ and $g_{\alpha\beta}(\xi^\alpha, \tau)$ and by the choice of local inertial tetrads

S. We note here that, in order to compute kinematic mechanical characteristics for the absolute velocity and acceleration vectors, of components of strain tensors, and of strain rates in fixed Riemannian spaces, one has to introduce a set of locally possible defined inertial tetrads S at each point.

On the basis of the form of the metric and the family L, one defines the concepts of absolute four-dimensional velocity $\mathbf{u}(\xi^i)$ and absolute acceleration $a(\xi^i)$ by the formulae

$$\mathbf{u} = d\mathbf{s}/d\tau, \qquad \mathbf{a} = d\mathbf{u}/d\tau \tag{2}$$

In the coordinate system ξ^α, τ, the components of the four-dimensional velocity $\mathbf{u}$ satisfy the formulae $u^1 = u^2 = u^2 = 0$ and $u^4 = c$, while covariant components of these vectors in the metric (1) may be defined by $\mathbf{u}_\alpha = cg_{\alpha k}\mathbf{u}^k = cg_{\alpha 4}$ and $\mathbf{u}_4 = c$, so that $\mathbf{u}^k\mathbf{u}_k = c^2$.

On the basis of these formulae, the absolute acceleration of individual material points, like the acceleration relative to local inertial tetrads, may be written as

$$a = a_\alpha э^\alpha = \left(\frac{du_\alpha}{d\tau} + u_s\Gamma^s_{\alpha 4}\right)э^\alpha = c\frac{\partial g_{\alpha 4}(\xi^\alpha, \tau)}{\partial \tau}э^\alpha, \quad \alpha_4 = 0 \tag{3}$$

It is well known from practice in the construction of geometrical pseudo-Riemannian models of space and time that, as when modelling various media and fields, one can—and indeed must—introduce different models of four-dimensional spaces for pseudo-Riemannian spaces. In that connection great success has been achieved in applications and in further refinement of the concepts of space and time.

The model of Newtonian mechanics in which the geometrically defined coordinates form a three-dimensional part of Euclidean space, while the fourth (time) variable is τ, which is measured by synchronized clocks, is absolutely independent of any objects and events.

Further complication of the concept of space as a physical object involves a far-reaching extension, by introducing the four-dimensional metric (1) of pseudo-Riemannian spaces; in such spaces, by definition, choosing a suitable system of coordinates x^α, τ in an infinitesimal elementary volume, depending on the point M under consideration, one can express the metric, apart from higher-order quantities, as

$$ds^2 = c^2d\tau^2 - dx^{12} - dx^{22} - dx^{32} \tag{4}$$

For the corresponding element of the coordinate tetrad S with relative increments dx^1, dx^2, dx^2, $dx^4 = d\tau$, for which, by definition, the Christoffel symbols vanish ($\Gamma^k_{mn} = 0$), the tetrad S for the variables x^α and τ at the point M forms a local inertial frame of reference.

As is well known, if L^* is any curve in a Riemannian space, then, by a suitable transformation of coordinates to certain variables which depend on the form of L^*, one can introduce variables x^α and τ in such a way that the Riemannian space metric will be expressed as (4) in an infinitesimal neighbourhood of L^* not only at an isolated point of space M but at all points of L^*.

Such a system of coordinates attached to L^* is called a Fermi frame of reference; it is inertial on L^*. In the general case, one cannot transform the metric to the form (4) for points of finite volumes or finite areas of three- and two-dimensional surfaces. If the metric of a Riemannian space can nevertheless always be reduced to the form (4) simultaneously at all points of the space, this special case is called Minkowski space, which is similar to Newtonian space with a suitable three-dimensional Cartesian system of coordinates.

In the general case, the components of the metric tensors g_{ij} for the tensors of pseudo-Riemannian spaces depend on the absolute time τ, and therefore, as will be shown below, it is impossible to introduce global inertial systems. This is true, in particular, for metrics corresponding to Weyl spaces, which differ from Minkowski space and have Ricci tensor $R_{ij} = 0$, while their Riemann tensor coefficients satisfy the equalities $R_{ijkl} = W_{ijkl}$, i.e. they are equal to the components of the Weyl tensor.

Basis tetrads S may always be introduced at different points of Riemannian spaces for appropriate local metrics (4) allowing for Lorentz transformations; and this may be done in a universal way by using the local systems of inertial Cartesian systems of coordinates x^α, τ in the three-dimensional Euclidean spaces of Newtonian mechanics or in Minkowski spaces with different global metric forms (1) in the variables x^α, τ, provided these forms correspond to metrics of topologically equivalent Riemannian spaces. The tetrads S are holonomic in Euclidean and Minkowski spaces, non-holonomic in Riemannian spaces of the most general form.

Since global inertial systems of coordinates cannot exist in Riemannian spaces, it follows that, corresponding to the inertial local tetrads S in Cartesian coordinates in Minkowski spaces and in the corresponding systems of coordinates (x^α, τ) in Riemannian spaces, in variables x^α and τ we will have non-holonomic inertial tetrads as frames of reference. Such a frame of reference may be considered as an immediate, natural extension of the global inertial frame of reference in Minkowski spaces. The tetrads with x^α and τ will be the inertial frames of reference selected in order to introduce and locally define characteristic quantities when one is formulating fundamental postulates—axiom systems in small objects. The introduction of model invariant relationships in Minkowski tangent spaces may provide the basis for a global definition of this kind of relationship in Riemannian spaces, agreeing with those properties of the spaces that are defined in the small; this is the essence of pseudo-Riemannian spaces.

On the other hand, for any fixed system of coordinate lines of families L for the variable τ, any determination of the four-dimensional velocity and absolute velocity vectors yields results which, irrespective of whether the tetrads are non-holonomic, are invariant with respect to the different local inertial tetrads S. To compute them, therefore, one can use any local inertial tetrads S, holonomic or not. However, if one is determining, say, the components of strain-rate tensors and many other characteristic values at the points of the system L, the results will depend on the choice of a system of non-holonomic local tetrads S.

In the mechanics of continuous media, non-holonomic inertial frames of reference S may be singled out in a Riemannian space with the aid of Cartesian systems of tetrads S in the fixed tangent Minkowski spaces at all points of the time coordinate lines L, as comoving frames of reference in Riemannian spaces.

In addition, it is obvious that if the pseudo-Riemannian space is fixed, the components of the metric in canonical form (1) in the comoving system of coordinates may be changed by specifying a family of coordinate lines for the variable τ and applying transformations

$$\xi^\alpha = \varphi^\alpha(\eta^\beta), \quad \tau = \tau' + \psi(\xi^1, \xi^2, \xi^3) \tag{5}$$

that preserve the form of the metric (1).

On given lines L, obviously, the absolute four-dimensional velocities and accelerations are independent of the form of the inertial tetrads of the reference system. However, such characteristics of single points as the strain tensor and strain-rate for small three-dimensional volumes dV_3, normal to the lines L depend on the frame of reference of local non-holonomic inertial tetrads S applicable to each point of the lines L, since these tetrads are moving at constant three-dimensional velocities $\mathbf{V}_{rel}$ along L. In any fixed four-dimensional Riemannian space, in suitable coordinates, proper choice of tetrads S and application of transformations (5) will make it possible to introduce simplified versions of the values of the metric tensors, for which the three-dimensional strain-rate tensors vanish.

Indeed, let $Э_1$, $Э_2$, $Э_3$ and $Э_4$ be basis vectors depending on time τ, at the points under consideration on the τ-coordinate lines of the comoving frame of reference, and suppose that at the points on the lines L we have constant basis vectors e_1, e_2, e_3 e_4 (independent of τ) for a local inertial tetrad S; assume that $Э_4(\tau)$ and e_4 at the points in question are directed along the corresponding tangent to a line of L.

If the inertial tetrads S at all points of the space are chosen in a non-holonomic way locally so that $Э_i = e_i$, then the set of inertial tetrads may be considered as a frame of reference. At the same time, we can write

$$g_{\alpha\beta} = (\ni_\alpha \ni_\beta) = (e_\alpha e_\beta), \quad g_{\alpha 4} = (\ni_\alpha \ni_4) = (e_\alpha e_4) \tag{6}$$

Formulae (6) define the same components of the metric tensors, but with possibly different frames of reference—this may, however, affect their derivatives with respect to global time τ.

Obviously, $\partial g_{\alpha\beta}/\partial\tau = 0$ relative to the tetrad frame of reference as described, with bases e_α, and so

$$g_{\alpha\beta} = g_{\alpha\beta}(\xi^1, \xi^2, \xi^3) \tag{7}$$

Therefore the strain-rate tensor, which is defined in terms of $g_{\alpha\beta}$, will vanish; this is quite satisfactory for motion along L in the theory of the free motion of material points, dust particles or even finite bodies which are absolutely rigid in gravitational theory.

On the basis of (2) and (3), the derivatives $c\partial g_{\alpha 4}/\partial\tau$ are the components of the acceleration relative to the tetrad frame of reference.

The previous definitions and terms have a purely kinematical nature in four-dimensional pseudo-Riemannian spaces.

To explain the applications of these theories to the GTR, let us write down the fundamental equations in a somewhat unusual but explicit form in CGS units

$$R_{ij} - g_{ij}R/2 = 2.07 \times 10^{-48} T_{ij} \tag{8}$$

This implies

$$\nabla_i T^i_j = 0$$

which hold in any system of coordinates, in particular, for comoving systems with a metric in the form (1).

Equation (8) is obtained by inserting the value of the coefficient

$$k = 8\pi G/c^4 = 2.07 \times 10^{-48}\ \mathrm{s}^2/(\mathrm{g\ cm}) \tag{9}$$

as the gravitational constant in the GTR, expressed in terms of the scalar characteristic of the spaces, c and Newton's gravitational constant G.

The components T_{ij} correspond to the "energy-momentum tensor", which is needed to formulate additional postulates to substantiate various model media and fields, and which, for the case of pure gravitation in the GTR for free motion of masses, is postulated by the equation $T_{ij} = \rho u_i u_j$. Hence

$$T^i_i = \rho u_i u^i = \rho c^2 \tag{10}$$

where ρ is the rest mass density of the particles of the medium.

It is obvious that when constructing physical models (which, by nature, are always approximate) one must pay attention to the very small value of the coefficient k and the comparatively small value of the right-hand side, when, according to (10), $|\bar{\mathbf{u}}| = c$, the density ρ may possibly be extremely large inside stars (in reality, with finite values); nevertheless, the right-hand side is still an admissibly negligible small number.

Successful applications of GTR in astronomy have been achieved, nonetheless, because of the well-founded assumption that, instead of realistically modelling bodies with moderately large densities ρ of distributed moving masses, celestial bodies are treated as singular material points with colossal masses. In products involving super-small values of the coefficient k, these masses may produce admissible types of singular points for the right-hand side of Eqs (8), enabling one to work with complicated Riemannian spaces "containing singular points". In practical problems of mechanics, the invariant quantities $g_{ij}T^{ij} = T^i_i$ usually have very modest values, and therefore the right-hand side of (8) may prove to be beyond the limits of the modelling accuracy.

Thus, in many problems, it proves possible to omit the right-hand side of Eq. (2) in many regions of space, as a quantity outside the range of the possible or necessary accuracy of a useful theory. Nevertheless, mathematically or physically speaking, one can consider Riemannian spaces for which it is true that $R_{ij} - g_{ij}R/2 = 0$, and this implies that the Ricci tensor vanishes

$$R_{ij} = 0 \tag{11}$$

and may be of interest not only for purely scientific research, but also from practical points of view in theoretical applications of physics.

In particular, physics has long been based on the STR, and developed on that basis, using Minkowski space, which is a particular solution of Eqs (11). (The author has at his disposal a construction of the class of all solutions of Eqs (11), which is tremendous.)

Special consideration will now be given to the possibility of introducing global inertial frames of reference in Riemannian spaces in the case of canonical metrics of type (1).

By definition, inertial systems in small volumes are characterized by all the specific properties of Cartesian coordinates in pseudo-Euclidean spaces, for which the coordinate lines L are straight lines or geodesics in Riemannian space, non-intersecting adjacent lines in the small being parallel up to infinitesimals of more than the first order. This means that, up to infinitesimals of the first order, the normal distances between adjacent fixed coordinate lines are the same.

It should also be recalled that diverging or converging systems of straight lines (geodesics) in the small cannot serve as global coordinate lines in inertial frames of reference. In other words, if we let e_i $(i = 1, 2, 3, 4)$ denote the coordinate vectors of bases for inertial tetrads, these vectors are approximately constant near the point in question.

After letting **u** and **a** denote the velocity and acceleration along all the coordinate lines L in inertial systems in the GTR in local tetrads

$$e_i u^i = dr/d\xi^i \approx \text{const}, \quad \mathbf{a} = 0$$

In four-dimensional Riemannian spaces in local inertial tetrads, the following relationships must hold exactly for the vectors **u** and for bases e_i and e^i, respectively, in any frames of reference (systems of coordinates)

$$\nabla_i u^k = 0 \ \text{ for any } \ k, i = 1, 2, 3, 4 \tag{12}$$

These relationships constitute a mathematical definition of inertial tetrads.

In Eq. (12), **u** is the basis vector corresponding to the time coordinate τ in the comoving metric (1). For inertial tetrads, however, the same Eq. (12) will hold by definition for each of the basis vectors $e_4 = \mathbf{u}$.

The invariant equation (12) may be applied in any system of coordinates.

The system of bases e_1, e_2, e_3, $e_4 = \mathbf{u}$ is not necessarily orthogonal!

At the same time, Eq. (12) is invariant under any transformations $u^*_p = l^i_p u^p$, where l^q_p are constant coefficients; this is true, in particular, for the transformation from covariant vectors e_p to contravariant vectors e^p.

Equation (12) may be rewritten in any system of coordinates, including a comoving system, as

$$\frac{\partial \overset{*}{u}{}^k}{\partial \xi^i} + \overset{*}{u}{}^p \, \Gamma^k_{pi} = 0 \tag{13}$$

It follows from (12) and (13) that

$$\Gamma^k_{4i} = \frac{1}{2} g^{kq}\left(\frac{\partial g_{q4}}{\partial \xi^i} + \frac{\partial g_{iq}}{\partial \tau} - \frac{\partial g_{4i}}{\partial \xi^q}\right) = 0 \text{ and } g_{k3}\Gamma^k_{4s} = 0$$

and, therefore, in global comoving inertial frames of reference one obtains

$$\frac{\partial g_{q4}}{\partial \xi^i} - \frac{\partial g_{4i}}{\partial \xi^q} = 0 \text{ and } \frac{\partial g_{iq}}{\partial \tau} = 0, \quad i, q = 1,2,3,4 \tag{14}$$

By (14), the Riemannian metric in the comoving system reduces to a synchronous form

$$ds^2 = c^2 d\tau^2 + g_{\alpha\beta}(\tilde{\xi}^\gamma) d\xi^\alpha d\xi^\beta = c^2 d\tau^2 - [\hat{g}_{11}(\eta^\gamma) d\eta^{1^2} + \hat{g}_{22}(\eta^\gamma) d\eta^{2^2} +$$

$$+ \hat{g}_{33}(\eta^\gamma) d\eta^{2^2}], \quad \gamma = 1,2,3 \tag{15}$$

Therefore, the world lines L are parallel geodesics and normal to a three-dimensional Riemannian surface Σ moving along the lines L at a constant four-dimensional translational velocity $\mathbf{u}$, which is the same at all points of the lines L.

If it is true, apart from the inertial property in Riemannian space (the case of empty volumes), that the corresponding components of the Ricci tensor vanish ($R_{ij} = 0$), then the three-dimensional space Σ is Euclidean and the metrics (6) define Minkowski space.

In the general case the components of the metric tensors g_{vj}—the tensors of pseudo-Riemannian spaces for arbitrary comoving families of lines L—may depend essentially on the absolute time τ. It is therefore impossible to introduce global inertial systems. In particular, it will be shown below that for metrics of Weyl space, which differ from Minkowski space in which also $R_{ij} = 0$, no global inertial system of coordinates can exist. Hence conditions (14) cannot hold globally in finite volumes.

Thus, in Riemannian spaces, one cannot generally have finite volumes V_4 with a global inertial system.

At every point of the space, locally applicable inertial tetrads for calculating the acceleration on a given world line, as a comoving system, are determined not uniquely but only up to a Lorentz transformation. Since the absolute acceleration vectors at points of world lines L in tetrads related to one another by a Lorentz transformation are the same, we may assume that the tetrads are the same at different points of L and on different L'. Hence one can use a locally identical unique fixed inertial tetrad at all points of Riemannian space.

There is a one-to-one correspondence between different topologically equivalent Riemannian spaces under which the same coordinate values correspond to different metrics. In particular, Cartesian coordinates and comoving inertial tetrads can be introduced by using Minkowski spaces and suitable inertial tetrads, which are also preserved in pseudo-Riemannian spaces but form a non-holonomic frame of reference in a Riemannian space.

Every finite relation between tensor components, as expressed in each tetrad, maintains its tensor form in the transformed components after suitable transformations of the systems of coordinates, both in Minkowski space and in Riemannian space.

This situation is the basis for convenient definitions of suitable tetrad systems in finite volumes, obtained through transformations for a fixed global metric.

As regards the inverse problem—to determine global metrics and the corresponding finite tensor relations in global coordinates on the basis of tetrad data—this requires the satisfaction of continuity conditions of a fairly general kind, taking into account that any fourth-rank tensor with suitable symmetry, in any coordinates—global or inertial local tetrad—may be considered as a Riemann tensor if its components satisfy the Bianchi identities only locally in the tetrads.

In fixed spaces with coordinates x^i, the transformations may be used to define the laws of motion of individual geometrical points or of individual physically defined points of continuous media or fields, for which the coordinates may or may not coincide with definite individual geometrical points of four-dimensional spaces. If one admits the application of a suitable collection of non-holonomic local inertial tetrads as frames of reference in Fermi variables x^α and τ, one can ensure satisfaction of the equality

$$\partial g_{\alpha\beta}/\partial\tau = 0 \quad \text{for} \quad \alpha, \beta = 1, 2, 3 \tag{16}$$

in comoving coordinates in four-dimensional pseudo-Riemannian spaces at every point and on every coordinate line L, where τ is an invariantly defined proper time.

If the right-hand sides of Eqs (9) are obtained by varying invariant energy expressions for a space of distributed masses, one can write at every point

$$k\left[\rho g_{ij}u^i u^j + \rho\frac{dU(x^\alpha,\tau)}{d\tau}\right]dV_3 d\tau = kT_s^s dV_3 d\tau = dm(g_{ij}u^i dx^j + dU) \tag{17}$$

where $dV_3 d\tau = dV_4$, with dV_3, $d\tau$ and dV_4 considered as invariant quantities. Then one obtains the same fundamental equation of field theory for the geometry of the Riemannian space, irrespective of the presence or absence of a specific potential energy $U(x^\alpha, \tau)$ (the equation is obtained by varying the energy equation with respect to g_{ij}). This equation, as is well known, occurs in the GTR as a condition for determining the geometrical properties of pseudo-Riemannian spaces as sets of points of a stock of mechanical objects and events associated with them.

By (17), the equations of field theory in any global or local tetrad frames of reference may be written in the form

$$R_i^j - \frac{1}{2}\delta_i^j R = k\rho u_i u^j \tag{18}$$

(The details of a dynamical relativistic theory of gravitation were presented by the author in a previous publication [2].)

Tensor contraction relative to i, j in Eqs (18), in any system of coordinates, yields the invariant formula

$$-R = k\rho c^2 \tag{19}$$

Next, covariant differentiation of Eqs (18) in any comoving coordinates yields

$$u_i\nabla_j(\rho u^j) + \rho u^j\nabla_j u_i = 0 \quad \text{or} \quad u_i\nabla_j(\rho u^j) + \rho c\nabla_4 u_i = 0$$

In a comoving system of coordinates the quantities $c\nabla_4 u_i$ are equal to the component a_i of the absolute acceleration

$$a_\alpha = c\partial g_{\alpha 4}/\partial\tau \tag{20}$$

on any individual comoving line L. Moreover, the acceleration projects onto the tangent to the trajectory in question as zero, because the absolute value of the velocity is constant: $|\bar{\mathbf{u}}| = c$.

In the general case, for an arbitrary comoving line, it is not necessary that a three-dimensional individual physical particle should have constant mass, i.e.

$$\nabla_j\rho u^j \neq 0 \quad \text{and} \quad a_i = \nabla_4 u_i \neq 0 \tag{21}$$

The law of conservation of mass for a particle on a line L implies the equation of continuity $\nabla_j\rho u^j = 0$, and therefore $a_i = 0$ and

$$\mathbf{a} = 0. \tag{22}$$

The theory of gravitation is constructed with due attention to two independent axioms, which represent empirical laws: the law of conservation of mass for individual particles, and the law of universal gravitation. These laws are established separately and independently of Eqs (9).

The potential energy U, as a function of the points of space, is the same for different comoving families in fixed spaces and, generally speaking, is determined using universal scalar differential equations in any fixed systems of coordinates; these equations do not depend on the metric of the Riemannian space in question.

The formulation of each of these laws consists of empirically justified model postulates; these naturally admit of different formulations of the characteristic magnitudes of the masses involved, also including various empirical constants. The formulae obtained may differ, depending on the desired modelling accuracy.

At points of vacuous volumes, where $T_{ij} = 0$ or $\rho = 0$, it is true that $R = 0$ and $R_{ij} = 0$, but the components of the Riemann tensor are generally non-zero and equal to the components of the Weyl tensor W, whose components are $W_{ipjq} = R_{ipiq} \neq 0$, contraction giving $W_{ip^{\cdot}q} = W_{ipj^{p}} = 0$. The algebraic properties of the components of the Weyl tensor were analysed by A. Z. Petrov, and their global functional properties have been established by the present author [3].

In a general setting the Weyl tensor corresponds to curved pseudo-Riemannian spaces depending on four scalar invariants.

By definition, Minkowski space and the STR are based on the simplest special case of the Weyl tensor, that is, zero.

Let us consider the solutions of the field equation (18), when the components of the metric in the canonical comoving coordinates depend only on the coordinates ξ^1, ξ^2, ξ^3. In the pure theory of gravitation in the GTR, for free motion of material points in canonical comoving coordinates, one deduces that in pseudo-Riemannian spaces the functions $g_{\alpha 4}$ are independent of τ: $g_{\alpha 4}(\xi^\gamma) \neq 0$, which is a corollary of the general equations (9) when the energy-momentum tensor is defined by formula (17) with $U = \text{const}$ at all points of the space.

The form of the function $g_{\alpha\beta}(\xi^\gamma)$ is determined by the choice of non-holonomic local inertial tetrad reference systems, which must always be used according to the postulates of a pseudo-Riemannian space, since in pseudo-Riemannian spaces a global holonomic inertial system of coordinates exists only in Minkowski space, for which Eqs (9) are not satisfied in the GTR when $\rho \neq 0$.

The functions $g_{\alpha\beta}(\xi^\gamma)$ may be considered, via a transformation of the type $\xi^\alpha = \varphi^\alpha(\eta^\gamma)$ as the tensor metric of a moving fixed three-dimensional definite Riemannian space P, independently of the global time coordinate τ, which is responsible for its arbitrary motion in four-dimensional space as an absolutely rigid body.

The functions $g_{\alpha 4}(\xi^\gamma)$ ensure the absence of absolute accelerations for four-dimensional velocities $\mathbf{u}$ on the global time τ-coordinate lines. Therefore, in a comoving metric for Eqs (18) with $g_{\alpha\beta}(\xi^\gamma)$, the motion of the three-dimensional space P must be translational and inertial in a four-dimensional space, which is in general curved if $R \neq 0$ or also three-dimensionally curved if $R = 0$ and $R_{ij} = 0$. However, for a three-dimensional metric $g_{\alpha\beta}(\xi^\gamma)$ one concludes that $R_{\alpha\beta} \neq 0$ when the Riemann tensor is equal to the Weyl tensor.

We know that, if $R_{\alpha\beta} = 0$, in P then the three-dimensional space P will be Euclidean, and the four-dimensional space will then be Minkowski space.

At the same time, one can consider Weyl and Minkowski spaces with moving accelerations of rigid three-dimensional spaces with curved spaces P, but in that case the absolute accelerations of individual material points must be non-zero and it is therefore necessary that $g_{\alpha 4}(\xi^\gamma, \tau) \neq 0$, which is inconsistent with the classical field equation (18) in GTR.

The dependence of $g_{\alpha 4}$ on ξ^γ is stipulated as a direct and quite natural extension of the Newtonian theory of gravitation for relativity theory, when the absolute acceleration is zero.

In Weyl spaces $g_{\alpha 4}(\xi^\gamma, \tau)$ and $\mathbf{a}$ may be non-zero. This is true, in particular, in comoving frames of reference in Minkowski spaces in the STR, when the potential energy is such that $U \cdot dm \neq 0$ for various values of $U = \text{const}$ on different coordinate time lines L for a curvilinear global variable τ.

The above constructive analysis of solutions of the field equations (18) in comoving coordinates enables us to characterize Riemannian and Weyl spaces, which must satisfy the classical field equation. The construction may provide a basis for understanding why it is necessary to introduce a potential energy which depends on the mass distribution of matter. It is nevertheless admissible from a mathematical standpoint, in conformity with the GTR, to construct a model theory of gravitation without taking the potential energy of masses into account, bearing in mind that $U \ll c^2$. However, even for very small absolute accelerations, the global geometrical and temporal properties of orbits determine different orbits of individual points when $U \neq 0$ or, when $U = \text{const}\,(L) \neq 0$, for large time intervals.

For a rigorous and natural account of the transition to Newtonian theory as a limiting case of the GTR, complicated by conceiving of the physical nature of space in terms of pseudo-Riemannian spaces, one should construct a theory of gravitation taking the potential energy into account, which is of fundamental importance in Newtonian theory. This should be done within the framework of Eqs (9) for the "energy-momentum tensor", which is defined by (17) with the addition of physically very essential scalar laws for the function U, representing the law of universal gravitation. The advantages of such a theory are obvious in practice in many situations in which accelerations are generated by an interaction in the presence of large concentrated masses, e.g. in connection with the purported reality of the theoretical phenomena associated with "black holes", which are characterized by a singular structure of space geometry and also be very high accelerations of individual material particles; and there are many other examples.

What can be said, according to Newton, about the energy of a glass with stationary water in Lake Sevan, or about the energy measured by a cosmonaut in an orbital station, or in a Yerevan laboratory, in comoving coordinates or in global inertial systems of coordinates? The answer given in the GTR is: these energies—for water (or a light snowflake)—are the same in both cases, each equal to mc^2. In Newtonian mechanics, however, and in the alternative (17) theory of relativity, one says that these energies are different, because of the presence of potential energies. Practical engineers, exploiting these differences, build electric power stations.

According to the aforesaid, in gravitational theory one can separate the question of defining a global space associated with the equations of field theory from the definitions of spaces and solutions of physical problems involving the motion of individual particles.

In Newtonian mechanics and alternative theories it is useful to use stationary fixed spaces in which different gravitational fields and laws of motion for mechanical systems can be constructed as exact solutions in the proposed models. Such a formulation of the problem may always be used in Newtonian mechanics and the STR.

In defining the spaces one can bear in mind the following model definitions.

1. In Newtonian mechanics, three-dimensional spaces are Euclidean, and time is absolute for all possible problems of mechanics.

2. In the special theory of relativity, everything is the same as in Newtonian theory, but only in comoving coordinate systems. For given observers, further processing, by inertial navigation algorithms, is necessary.

As follows from the aforesaid, for arbitrary L or L^* the problem is that, unless suitable restrictions are imposed or additional conditions placed on the family of lines L, the fundamental three-dimensional equations (9) in the GTR admit of a large variety of exact model solutions of Eqs (8), which cannot serve as satisfactory models to describe the nature of gravitational fields.

In view of the nature of Eqs (8), particular solutions cannot be determined unless additional conditions are imposed. In the GTR these additional conditions amount to the requirement that the lines L be geodesics.

In Newtonian mechanics, the additional conditions comprise the validity of the well-corroborated experimental model law of universal gravitation for distributed masses of material bodies. These masses m, in turn, are defined (by empirical postulates) as characteristic parameters of bodies and are of fundamental value in all applications of mechanics to theories of the motion of material bodies.

In the GTR, many authors assume that the law of universal gravitation is automatically satisfied in weak gravitational fields. In gravitational theory, however, if the geodesicity condition is or is not satisfied for certain systems of comoving families L, then the same will hold after coordinate transformations for any other comoving families L' and the time variable τ'.

In Pauli's *Relativitätstheorie,* first published in 1921, which had a tremendous influence on the ideology of many authors of later scientific papers and textbooks on the GTR, one finds the following fundamental assumptions (see [10, pp. 203 and 219]).

In weak gravitational fields, the components of the metric tensor must deviate only slightly from the values corresponding at each point of space to a local inertial metric. Otherwise, basing himself on this assumption, Pauli proceeds from the metric that he postulates for Riemannian space

$$ds^2 = c^2\left(1 + \frac{2\Phi(x^1, x^2, x^3)}{c^2}\right)dt^2 + g_{\alpha\beta}(x^1, x^2, x^3))dx^\alpha dx^\beta \tag{23}$$

Pauli took a small function $\Phi(x^1,\ x^2,\ x^3)$ independent of the variable t, which is not satisfactory at all because of the meaning later given by Pauli himself to that function.

As additional conditions (which are always necessary), historically speaking, a particular, special metric of the form (23) was proposed as a satisfactory metric for the GTR; but this is inherently inadmissible if one is trying to verify the validity of the theory for weak fields, whenever the solutions of Eqs (9) of the GTR approach Newtonian theory.

One should immediately stress that any additional conditions in fact impose restrictions on the system of non-unique solutions of the fundamental equations (9) in the GTR, while the postulated forms of the metrics in question may not be solutions of the relevant problems at all, or they may fail to correspond to reality from a physical point of view.

We are going to prove that any solutions describing the free motion of particles of constant mass in spaces with metric (23) in comoving coordinates admit of absolute accelerations; but by the field equations (18) with $T_{ij} = k\rho u_i u_j$ this contradicts the main conclusion of the GTR, that the time orbits in material four-dimensional spaces are geodesics.

However—and this is true in particular for the Schwarzschild metric in the GTR, which differs from (23) and is also defined on the basis of a series of limited conditions—one can ensure the validity of the equations of the GTR and the geodesicity of the motion of the test particles. In that case one concludes, further, that the individual test particles move along geodesics, without interacting with adjacent particles, with constant energy values mc^2 in each orbit in a space whose Gaussian curvature is zero, $R = 0$, in vacuous regions.

Explicit definitions of individual points and invariantly defined proper time are necessary if one wishes to apply the concepts of absolute four-dimensional vectors of velocity and acceleration. To that end, besides the coordinates $(x^1,\ x^2,\ x^3,\ t)$ in (23), one introduces comoving Lagrange coordinates $\xi^i(\xi^1,\ \xi^2,\ \xi^3,\ \tau)$ via the following transformations

$$x^\alpha = x^\alpha(\xi^1, \xi^2, \xi^3, \tau), \quad \text{or} \quad dt\sqrt{c^2 + 2\Phi} = d\tau c \quad \text{or} \quad t = \int^\tau \left(1 + \frac{2\Phi}{c^2}\right)^{-1/2} d\tau \tag{24}$$

The metric (23) is reduced by the transformation (24) in the same space to the form

$$ds^2 = c^2 d\tau^2 + 2\hat{g}_{\alpha 4}(\xi^\gamma, \tau)d\xi^\alpha d\tau + \hat{g}_{\alpha\beta}(\xi^\gamma, \tau)d\xi^\alpha d\xi^\beta \tag{25}$$

After each transformation in the variables (24), we obtain a canonical comoving metric, relative to which the law of motion in Lagrangian coordinates (24) involves a global proper time $\tau \neq t$, e.g. one has "initial" data $\xi^\alpha = \text{const}$ on the trajectories of individual particles. We note, in addition, that metrics (25) may represent different laws of motion, depending on the choice of the functions $x^\alpha(\xi^\gamma, \tau)$, or, equivalently different definitions of transformations (24), fixed by removing the arbitrary specification of the functions $x^\alpha(\xi^\gamma, \tau)$ from the indicated arguments in formulae (25).

For series of special transformations, we set

$$x^\alpha = \frac{\xi^\alpha}{b}, \quad \left(1+\frac{2\Phi}{c^2}\right)^{-1/2} d\tau = dt \tag{26}$$

where b is any possible finite scalar constant. Based on formulae (23) and (26), one can write down the components of the metric in (25). This yields the approximations

$$\begin{aligned} g_{\alpha 4} &= c^2\left(1+\frac{2\Phi}{c^2}\right)\frac{\partial \tau}{\partial \xi^\alpha}\frac{\partial t}{\partial \tau} = c^2\left(1+\frac{2\Phi}{c^2}\right)^{1/2}\int^{\tau} \frac{\partial \tau}{\partial \xi^\alpha}\left[1+\frac{2\Phi(\xi^\alpha / b)}{c^2}\right]^{-1/2} d\tau = \\ &= c^2\left(1+\frac{2\Phi}{c^2}\right)^{1/2}\int^{\tau}\left(1+\frac{2\Phi(\xi^\alpha / b)}{c^2}\right)^{3/2}\frac{\partial \Phi}{\partial x^\alpha}\frac{1}{b} \approx -\frac{1}{b}\int^{\tau}\frac{\partial \Phi}{\partial x^\alpha}d\tau \\ \hat{g}_{\alpha\beta} &= \frac{1}{b^2}g_{\alpha\beta} + g_{44}\frac{\partial t}{\partial \xi^\alpha}\frac{\partial t}{\partial \xi^\beta} \approx \frac{1}{b^2}g_{\alpha\beta} - \frac{1}{b^2}\int^{\tau}\frac{\partial \Phi}{\partial \xi^\alpha}d\tau\int^{\tau}\frac{\partial \Phi}{\partial \xi^\beta}d\tau \end{aligned} \tag{27}$$

Using the formula $a_\alpha = c\partial g_{\alpha 4}/\partial \tau$ one sees that the components of the acceleration of individual particles on lines L corresponding to values $\xi^\alpha = \text{const}$ are defined by the formula

$$a_\alpha = -\frac{c}{b}\frac{\partial \Phi}{\partial x^\alpha} \tag{28}$$

According to (28), the components of the absolute acceleration vector **a** may be different, but it is essential that $\mathbf{a} \neq 0$.

This conclusion has been deduced for a special transformation (26); but the metric in another comoving frame of reference, corresponding to a general transformation, is derived from (25) by applying after (26) an additional transformation, preserving the form of the canonical comoving metric (25). It is easily shown that any such transformation is given by the formulae

$$\xi^\alpha = \varphi^\alpha(\eta^\gamma) \quad \text{and} \quad \tau = \tau' + \Psi(\eta^\gamma)$$

Transforming the metric (25) in this way, we obtain

$$\hat{\hat{g}}_{\alpha 4} = c^2\frac{\partial \Psi}{\partial \eta^\alpha}\frac{\partial \tau}{\partial \tau'} + \hat{g}_{\beta 4}\frac{\partial x^\beta}{\partial \eta^\alpha} - 1 + (-1)0 \tag{29}$$

Since $c^2\partial\Psi/\partial\eta^\alpha$ is independent of τ, it follows that in other comoving coordinates η^α, τ' one has the following formula for the non-zero acceleration

$$\hat{\hat{a}}_\alpha = \frac{c\partial \hat{\hat{g}}_{\alpha 4}(\eta^\gamma, \tau')}{\partial \tau'} = \hat{a}_\beta\frac{\partial x^\beta}{\partial \eta^\alpha}, \quad \beta = 1,2,3 \tag{30}$$

The components a_α differ from a_β only through the transformation from ξ^α, τ to η^β, τ; hence the absolute acceleration vector is an invariant.

Since L is an arbitrary comoving line for the free motion of a particle of constant mass, it follows from (22) in the GTR, with potential energy $U = \text{const}$ at all points of space, that the time acceleration $\mathbf{a}$ is necessarily non-zero in any comoving coordinates of the metric (23), and therefore it does not satisfy the fundamental condition of the GTR and is not a solution of any problem involving the free motion of material media in the GTR.

It is clear from (27) that the components $g_{\alpha\beta}(x^\alpha, \tau)$ do not affect the magnitude of the acceleration $\mathbf{a}$ and therefore the metric (23) cannot be chosen so that $\mathbf{a} = 0$; but one can choose $g_{\alpha\beta}$ so that Poisson's equation is only approximately satisfied, on the basis of the approximate equality (19) for the metric (23)

$$R = -k\rho c^2$$

and after introducing the constant relativistic gravitational coefficient k of formula (9). In view of the very small values of k, the accuracy in the values of the coefficient obtained through what are actually "false" solutions in the GTR is doubtful, but it is quite satisfactory when allowance is made for the potential energy, which is the source of the observed accelerations.

Thus, the metric (23) is not consistent with the GTR, in which $T_i^i = dmc^2$ and $\mathbf{a} = 0$. If one assumes that $T_i^i = dmc^2 + dmU$, then instead of the equality $a_\alpha = 0$ one obtains $a_\alpha = -\partial U/\partial x^\alpha$ and $U = \Phi$.

We add here that the quantity k does not influence the solutions of the equations of the GTR in vacuous volumes for Weyl spaces, which are defined by the equality $T_{ij} = 0$, when, however, the density is infinite, $\rho = \infty$, only at isolated points of space where the mass m is finite (the model of a star).

In Newtonian field theory and alternative relativistic theory with potential energy, when equality (17) holds in the comoving frame of reference and the scalar Poisson's equation is introduced independently of the system of equations (8), in which it follows from the law of universal gravitation that

$$\nabla^\alpha \nabla_\alpha U = -4\pi\rho G \tag{31}$$

the constant k will be replaced in solutions by the gravitational constant G as derived from Eq. (31) and not from the canonical equations (9) in the GTR, since in a vacuum $T_{ij} = 0$. However, even in an approximate formulation one cannot ignore the non-vanishing acceleration $\mathbf{a}$ of (28), since if one puts $\mathbf{a} = 0$, so that, by (28), $\Phi = \text{const}$, then the metric (23) will define Minkowski space, in which all geodesics are straight lines, in glaring contradiction to the main corollaries of physical gravitational theory.

Nevertheless, in short time intervals, one can take $\tau = t$ after suitable approximations; in that case the planets move relative to the Sun with small accelerations and therefore, in practice, the terrestrial frame of reference may often be treated as inertial in many problems (but as a rule taking into account the accelerations of forces of gravity generated by the Earth); in addition, of course, no allowance need be made for the alteration of day and night!

At the same time, Pauli took it for granted that the law of universal gravitation and various refinements thereof were valid in the GTR, and accordingly stated Poisson's equation followed from the equations of the GTR for very weak fields. However, the metric (23), which he proposed on intuitive grounds, is an additional strong assumption, which explicitly contradicts the required geodesic property of orbits in the GTR even for weak fields.

Similar assertions may be found in a large number of scientific publications and textbooks on the GTR. In particular, it is claimed that in celestial mechanics the GTR yields the next amendment to Newtonian theory in general, including the case of long time intervals, though this has not really been proved.

Thus, the previous and subsequent considerations contradict Pauli's conclusion. Here is a quotation from Pauli's text: "Thus, Poisson's equation indeed turns out to be valid. The fact that the general theory of relativity, based on the postulates of §56 without further assumptions, leads to Newton's law of gravitation is its greatest success. Moreover, as a result we are now in a position to say something about the sign and the numerical value of k."

In fact, the geodesic property of the family of lines L in the GTR follows precisely from the condition that the components $g_{\alpha 4}$ be independent of the proper time τ, in other words, only as a result of the form of the function $u_\alpha(\xi^1, \xi^2, \xi^3)$ and accordingly in the comoving system of coordinates when $g_{\alpha 4}(\xi^\gamma)$.

The possible freedom in the choice of the family of coordinate lines L and the components $g_{\alpha 4}(\xi^\gamma)$ and $g_{pq}(\xi^\gamma, \tau)$ shows that one can construct the gravitational field in more than one way even in vacuous regions, where $\rho = 0$, when there are singular points or boundary conditions for the sets of pseudo-Riemannian spaces for some series of families of time lines L with

$$R_{ij} - \frac{1}{2} g_{ij} R = 0.$$

This kind of solution is certainly of little use, generally speaking, for models of gravitational fields in nature, without taking into account the interaction of masses according to the law of universal gravitation.

It is nevertheless obvious that explicit influence of the constant k is excluded in the fundamental equations for vacuous volumes.

In alternative theories of relativity and in Newtonian theory one assumes that the specific potential energy U satisfies Eq. (31), according to which absolute accelerations do not vanish, while in the dynamical theory one proves that

$$\mathbf{a}_{\text{abs}} = - \operatorname{grad} U \tag{32}$$

In that case the metric (23) is a solution of Eqs (8) with the right-hand sides modified by introducing the specific potential energy. The point is that in relativistic alternative theories the geodesic orbits of the GTR are replaced by exact solutions with accelerations, just as in Newtonian theory, but in pseudo-Riemannian spaces. When that is done the continuity of the passage to Newtonian mechanics as a limit is natural and mathematically rigorous.

If the family of lines L is given, then by (32) one can define a scalar function U in terms of the acceleration $\mathbf{a}_{abs}$. It is also obvious that the converse is also true, i.e. the lines L can be determined, given U. It is therefore clear that in order to obtain the necessary models for a correct description of gravitation in nature, generally speaking, the use of the model law of universal gravitation is essential, independently of Eqs (8)!

The laws of the corresponding motions of individual points in Riemannian spaces are generally different, and this is essential for evaluating theories, since it is these laws that determine the main properties of orbits of celestial bodies over large time intervals and in weak gravitational fields.

Let us consider Schwarzschild theory for the solutions of Eqs (8). Instead of the metric form (23) we now take a Schwarzschild space metric, which is generated by a polar system of space coordinates r, θ, ψ, c with time coordinate t in Newtonian mechanics, or in the GTR for a pseudo-Riemannian space with a Schwarzschild metric satisfying Eqs (8) in vacuous volumes of four-dimensional space. We can also take spaces filled with "small test masses", when the approximate formulation of the mathematical problems involved is such that the influences of these masses on the geometry may be disregarded, hence also their influence on the Schwarzschild metric, which is defined by one singular material point with a large concentrated mass M, and models a stationary three-dimensional metric of four-dimensional pseudo-Riemannian space (ignoring local fields in problems concerning the motion of celestial bodies that disturb the surrounding regions of space).

In the case of the Schwarzschild solution for Eqs (8) in the GTR, it turns out that the Gaussian curvature R is zero at all points of four-space except for the singular point. Therefore the Riemann tensor becomes equal to the Weyl tensor, corresponding, in particular, to a metric expressed in the following form, due to Driest and Weyl

$$ds^2 = c^2\left(1-\frac{r^*}{r}\right)dt^2 - \left(1-\frac{r^*}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\psi^2) \quad (33)$$

In this metric the only constant parameter characterizing the space, other than c, is r^*. In applications r^* is extremely small.

When $r^* = 0$ the metric (33) corresponds to Minkowski space, in which all geodesics are straight lines. Hence, considering planetary motion in the GTR, one cannot assume that $r^*/r = 0$ globally in approximate solutions.

It is thus obvious that in the GTR, even for very weak fields, the space depends invariantly only on the variable ratio r^*/r in the circular and non-circular orbits L under consideration. Moreover, in fixed Schwarzschild spaces the quantity r^* for the singular point at which $r = 0$ depends on the constant numbers M, c and G. The latter essentially exert their influence only through the combination $2MG/c^2 = r^*$, which in the GTR is an empirical magnitude depending on the specific situation.

We also note that the small variable quantities r^*/r cannot be omitted in orbits L in the metric (33), since it is allowance for these small variable ratios that causes the relativistic effects frequently treated in the GTR as refinements of Newtonian mechanics.

Let us consider the system of lines L in variables $t, r, \theta, \psi = \omega_0 ct$ with constant values of the Lagrange coordinates; assume that the following equalities hold on the system of time-coordinate lines L: $r = r_0 = \text{const}$, $\theta = \theta_0 = \text{const}$, $\omega = \omega_0 = \text{const}$.

Considering families of concentric circular orbits with different values of r_0 and ω_0 at $\theta = \theta_0 = \pi/2$ and $\psi = c\omega_0 t$, we can rewrite the metric (33) as

$$ds^2 = c^2\left(1-\frac{r}{r^*} - \omega_0^2 r^2\right)dt^2 - \left(1-\frac{r^*}{r}\right)^{-1} dr^2 \quad (34)$$

Instead of the time coordinate along L, i.e. the variable t, we can introduce a variable τ on L as a global coordinate for proper time on the lines of L, by applying the following coordinate transformation

$$\tilde{r} = r, \quad d\tau = \left(1-\frac{r^*}{r} - \omega_0^2 r^2\right)^{1/2} dt \quad \text{or} \quad t = \tau\left(1-\frac{r^*}{r} - \omega_0^2 r^2\right)^{-1/2} \quad (35)$$

As a result we obtain a solution of the field equation (9) in "three-dimensional" comoving coordinates

$$ds^2 = c^2 d\tau^2 + 2g_{14} d\tilde{r} d\tau + g_{11} d\tilde{r}^2 \quad (36)$$

where

$$g_{14} = \hat{g}_{44}\frac{\partial t}{\partial \tau}\frac{\partial t}{\partial r} = -\tau c^2(r^*/r^2 - 2r\omega_0^2)/2(1 - r^*/r - r^2\omega_0^2)^2$$

$$g_{11} = \hat{g}_{11}$$

By the fundamental law of the GTR, the absolute accelerations must vanish in the orbits L of freely moving individual mass particles. If the orbits L in Schwarzschild space are defined by the condition that the radial coordinate equal a constant, i.e. $r = r_0 = \text{const}$, then by (36) it is necessary and sufficient that

$$\omega_0^2 = r^*/(2r_0^3) \quad (37)$$

and then

$$\tau = t(1 - 3r^*/(2r_0))$$

If Eq. (37) fails to hold, then $g_{14} \neq 0$. Therefore the following formulae hold for the absolute accelerations

$$a_1 = c\frac{\partial g_{14}}{\partial \tau} = -\frac{c^3}{2}\frac{\partial}{\partial r}\left(1 - \frac{r^*}{r} - \omega_0^2 r^2\right)^{-1} = -\frac{\partial U}{\partial r}, \quad a_2 = a_3 = 0 \tag{38}$$

where the component a_1 does not vanish. In that case, considering the metric (34) in Schwarzschild space, one obtains the acceleration of the force of gravity with specific potential energy

$$U = -\frac{c^3}{2}\left(1 - \frac{r^*}{r} - \omega_0^2 r^2\right)^{-1} + U_0$$

which depends on two constants r^* and ω_0^2 and on the constant of integration U_0

$$r^* = \frac{2MG}{c^2}, \quad \omega_0^2 = \frac{\Omega^2}{c^2}$$

Thus, formula (38) for the acceleration component a_1 may be rewritten in the form

$$a_1 = -\frac{\partial U}{\partial r} = c\left(\Omega^2 r - \frac{MG}{r^2}\right)\left(1 - \frac{r^*}{r} - \omega_0^2 r^2\right)^{-2} \neq 0 \tag{39}$$

However, the conditions $a_1 = 0$ in the orbits $r = \text{const}$ in the GTR are equivalent to the analogous equality in Newtonian mechanics

$$\Omega^2 l = MG/l^2 \tag{40}$$

which may not be valid in a fixed Schwarzschild space in different orbits $r = \text{const}$, because, first, the Newtonian distances l of the points in the orbits from the point $r = 0$ are not equal to the Schwarzschild distances, $r \neq 1$; second, the angular velocity $\Omega = d\varphi/d\tau$ in the GTR does not equal its Newtonian counterpart $d\varphi/dt$, because $\tau \neq t$ as implied by (35). Therefore, if one replaces l by r in formula (40), considered in Schwarzschild space, the result is not the same as the corresponding Newtonian formula. The differences for test particles give rise to mechanical effects in the classical GTR.

It follows from formula (37) that in orbits $r_0 = \text{const}$, $\theta_0 = \text{const}$ and $\omega_0 = \text{const}$ on the corresponding lines L, the magnitudes of the time periods T_t and T_τ of rotation of the planets around the Sun are related by

$$T_\tau = T_t\left(1 - \frac{3}{2}\frac{r^*}{r_0}\right)^{1/2}, \quad T_\tau \neq T_t$$

It follows from (38) that if $r_0 = 3r^*/2$, then $T_\tau = 0$ and so the corresponding motion of the orbit occurs at a three-dimensional velocity equal to the speed of light c.

For very small $r^* \ll r$ the periods satisfy the following equalities with high accuracy

$$T_\tau = T_t\left(1 - \frac{3}{4}\frac{r^*}{r_0}\right), \quad T_\tau \approx T_t$$

Hence, when equality (37) holds, according to the GTR the test bodies—planets—in Schwarzschild space move forward along geodesics, in other words, $\bar{a}_{abs} = 0$ for their centre of mass.

However, if certain values of the constants r_0 and ω_0 do not satisfy Eq. (37), then the planets will not only move forward at $r_0 = \text{const}$ but also spin, as a result of which $\bar{a}_{abs} \neq 0$ for their centre of mass. Hence the orbits in Schwarzschild space in the GTR for rotating planets will not necessarily be geodesics in the theory of the metric (34).

We shall now consider exact formulae for the periods of rotation of small masses m around the Sun, whose mass M is assumed to be large, considering the motion as that of a material point describing circular orbits in Newtonian theory.

By the law of universal gravitation, the planet may describe a circle of radius $l = \text{const}$ under the action of the force of attraction F of the Sun, under the assumption that the Sun is stationary in the inertial frame of reference (the mass M is large and m small). The Newtonian forces of interaction may be written as

$$F = \frac{mMG}{l^2} = \frac{m\upsilon^2}{l} \Rightarrow \upsilon = \left(\frac{MG}{l}\right)^{1/2} \quad \text{and} \quad T_{\text{New}} = \frac{2\pi l}{\upsilon} = 2\pi l^{3/2}(MG)^{-1/2}$$

Similarly, using (34), we obtain the following formulae for the circular orbits of test masses in Schwarzschild space and in Euclidean spaces

$$r^2 = \frac{r^* c^2 T_t^2}{8\pi^2}, \quad l^3 = \frac{MG}{4\pi^2} T_{\text{New}}^2$$

By the definition established for Schwarzschild space, it is assumed that

$$r^* = 2MG/c^2$$

We can now write

$$\frac{r^3}{l^3} = \frac{T_t^2}{T_{\text{New}}^2} = (1-p)^3, \quad T_t = T_{\text{New}}(1-p)^{3/2} \tag{41}$$

For large r, p is small but essentially non-zero, since

$$r = (1-p)l \tag{42}$$

The only dimensionless parameter in the circular orbits being considered in the metric (33) is the quotient r^*/r, which is constant on different circles.

In the general formulation of the problem of stationary solutions as stated above, Schwarzschild found a family of solutions in the GTR which depend on two geometrical constant parameters $\alpha = r^*$ and ρ (in Schwarzschild's notation). For continuous solutions in polar coordinates, up to values $r = 0$ in a vacuum with $R_{ij} = 0$ and $R = 0$, Schwarzschild established the following relationship [11]

$$\rho = \alpha^3 = r^{*3}$$

and found a formula for the metric

$$ds^2 = c^2\left(1 - \frac{r^*}{R}\right)dt^2 - \left(1 - \frac{r^*}{R}\right)^{-1} dR^2 - R^2 d\Omega^3 \tag{43}$$

where the spherical coordinate element is $d\Omega^2 = d\theta^2 + \sin\theta d\psi^2$ and

$$R = r(1 + r^{*2}/r^3)^{1/3} \approx r$$

when there is no black hole, which may be considered as if collapsed into the singular point $r = 0$. The passage from the metric (43) to the metric (33), though seemingly quite legitimate, nevertheless qualitatively alters the geometrical structure of the pseudo-Riemannian space near the singular point.

If "ρ" = 0 and $\alpha = r^*$, the metric (43) takes the form of the metric (33) with all the consequences described above and, in particular, there is a black hole.

If "ρ" ≠ 0 and $\rho = f(r^*)$, where $f(r^*)$ is some given function, one obtains a series of solutions with different black holes.

This research was supported financially by the Russian Fund for Fundamental Research (93-013-17341).

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Translated by D.L.